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# Singularities in the Rayleigh-Taylor instability of a thin plasma slab

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A theoretical study of the interchangelike (Rayleigh-Taylor) instability of a thin slab of weakly ionized plasma is presented. We have found an analytical solution for the stationary motion of a plasma slab under the effect of the magnetic field pressure. This solution describes a shock-wave-like structure of the magnetic field and is unstable against the interchange mode. Using an approach developed earlier [E. Ott, Phys. Rev. Lett. **29**, 1429 (1972); S.V. Bulanov, and P.V. Sasorov, Sov. J. Plasma Phys. **4**, 418 (1978)] we have obtained exact solutions, in terms of analytical functions of a complex variable, of the Cauchy problem for the evolution of nonlinear perturbations. We have investigated the formation of the typical singularities that correspond to different wave breaking regimes in an unstable medium. We discuss Ott's problem of the Rayleigh-Taylor instability of initially nonplanar shells. [S1063-651X(99)10602-0]

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## I. INTRODUCTION

The Rayleigh-Taylor instability provides one of the best examples of the basic behavior of a fluid when its equilibrium configuration is unstable against infinitesimal perturbations. This instability plays an important role in many applications including inertial confinement fusion when the target is compressed by the laser light, plasma confinement in a magnetic field, and supernova explosions. In many cases the nonlinear features of the Rayleigh-Taylor instability can be studied analytically, since in various limits its evolution admits exact solutions. In the present paper we consider a model that provides a very transparent description of the nonlinear stage of the Rayleigh-Taylor instability. This model describes the dynamics of a thin slab of weakly ionized plasma under the pressure of a magnetic field.

Recently the magnetohydrodynamic behavior of weakly ionized plasmas has been the subject of several theoretical studies in space physics. These studies have been performed within the model of the coupled hydrodynamic equations for the ionized and for the neutral components [1–5]. The dynamics of weakly ionized plasmas plays an important role in space plasmas since they occur in protostellar disks, in the cores of molecular clouds [2] where stars form, and near the photosphere of the sun. Molecular clouds have a very low state of ionization, with ionization fractions around $\varepsilon=\rho/\rho^{(n)}\approx 10^{-7}$, where $\rho$ and $\rho^{(n)}$ are the ion and the neutral components of the plasma. Under typical conditions for space plasmas the magnetic field is as important as gravity in the molecular clouds [3], while, in the solar photosphere, magnetic field line reconnection can be invoked in order to explain bright point formation [4].

An important factor in weakly ionized plasmas, besides Ohmic and viscous dissipation, is the momentum and energy exchange between the ionized and the neutral components. This mechanism has been investigated in connection with the damping of magnetohydrodynamic (MHD) fluctuations in Refs. [1].

The effect of the ion-neutral interaction becomes important when the ion-neutral collision time $1/n^{(n)}v\sigma^{(in)}$ is much shorter than the typical time $\tau_0$ of the physical process under consideration. Estimating the cross section of the ion-neutral collisions as $\sigma^{(in)}\approx 5\times 10^{-15}$ cm$^2$, and the typical time $\tau_0$ as the Alfvèn time $\tau_a=l/v_a$, with $v_a$ the Alfvèn velocity, we can write the condition when the ambipolar diffusion due to ion-neutral collisions changes the regime of MHD motion as $l>2\times 10^{14}/n^{(n)}$ cm. Here $n^{(n)}$ is the neutral density, $l$ is the typical scale of the motion, and we have supposed that the Alfvèn Mach number $M_a=v/v_a\approx 1$. Under typical parameters of molecular clouds, where $l\approx 10^{18}$ and $n\approx 10^3$ cm$^{-3}$, this condition can be easily satisfied.

In the present paper, we present analytical solutions that describe stationary regimes of motion of a weakly ionized plasma in the magnetic field. We show that the magnetic field has a shock-wave-like structure. This configuration is unstable against the interchangelike instability. This instability is similar to the interchange instability of a fluid plasma supported against gravity by a magnetic field [6] and to the Rayleigh-Taylor instability. As is well known, the studies of the Rayleigh-Taylor instability are of great importance for inertially confined fusion since this instability is inherent to imploding plasmas and leads to the deterioration of the implosion symmetry [7]. The Rayleigh-Taylor instability in laser plasmas has been studied intensively both theoretically and experimentally (see Ref. [8] and references therein). We shall use the thin shell approximation developed in Refs. [9], [10], and [11] that allows us to give an analytical description of the nonlinear aspects of the Rayleigh-Taylor instability. We analyze the nonlinear stage of the interchange instability of a thin plasma slab of a weakly ionized plasma and discuss the typical structures of the singularities that appear as a result of the instability and that correspond to various regimes of the wave breaking in an unstable medium: rarefaction and compression wavebreak.

The thin shell approximation is equivalent to the study of the instability in the long wavelength limit which may become invalid in the final nonlinear stage of the instability. This limitation is discussed explicitly in Sec. VI, where it is

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argued that in the case of the rarefaction wavebreak the long-wavelength approximation remains valid up to the formation of the singularity itself. In the case of the compression wavebreak the long wavelength approximation ceases to be valid in the final nonlinear stage. Thus, in this case, the results presented in this paper have the meaning of ''intermediate asymptotics'' and describe the trend towards the formation of spatial singular structures.

## II. BASIC EQUATIONS: STATIONARY MOTION OF A PLASMA SLAB

In the weakly ionized plasmas in the long-wavelength approximation, which is supposed to be valid in the limit considered in the present paper, the momentum exchange between ionized and neutral components balances the Lorentz force and the force due to the plasma pressure gradient. Then, assuming in the limit $\rho^{(n)}/\rho \gg 1$ that the velocity of the neutral components is much less then that of the ionized component and neglecting the ion inertia, we obtain the system of equations:

$$\mathbf{v}=-\frac{1}{\nu^{(in)}}\left(\frac{\nabla P}{\rho}-\frac{(\nabla\times\mathbf{B})\times\mathbf{B}}{4\pi\rho}\right), \tag{1}$$

$$\partial_t\rho+\nabla\cdot(\rho\mathbf{v})=0, \tag{2}$$

$$\partial_t\mathbf{B}=\nabla\times(\mathbf{v}\times\mathbf{B}). \tag{3}$$

Here $\mathbf{v}$ is the velocity of the ionized component, $\rho$ is the plasma density, $\mathbf{B}$ is the magnetic field, $\nu=n^{(n)}v\sigma^{(in)}$ and $P$ is the plasma pressure. To these equations we must add the equation of state $P=P(\rho,T)=\rho T/m_i$. Here $m_i$ is the ion mass and $T$ is the sum of the ion and electron temperatures. We take these temperatures to be constant due to the high thermal conductivity and charge exchange rate.

In the one dimensional approximation, when all the functions depend on the $x$-coordinate and time only, Eqs. (1)–(3) have stationary solutions. These solutions describe a finite width plasma slab moving under the magnetic and plasma pressure with constant velocity $V$. We introduce the new variable

$$X=x-Vt \tag{4}$$

and obtain from Eq. (1)

$$\rho\nu^{(in)}V=-\frac{1}{m_i}\rho' T-\frac{1}{4\pi}B_zB_z'. \tag{5}$$

A prime denotes differentiation with respect to the variable $X$. The magnetic field has an $X$-dependent $B_z$ component frozen in the plasma which moves with constant velocity along the $x$ axis. We see that Eq. (5) gives one relationship for the two unknowns $B_z$ and $\rho$. This means that one of these functions is supposed to be given. The problem under consideration is characterized by two dimensionless parameters,

$$\alpha=\frac{\Gamma M}{\mathrm{Kn}}=\frac{V\nu^{(in)}\Gamma L}{c_s^2} \tag{6}$$

and

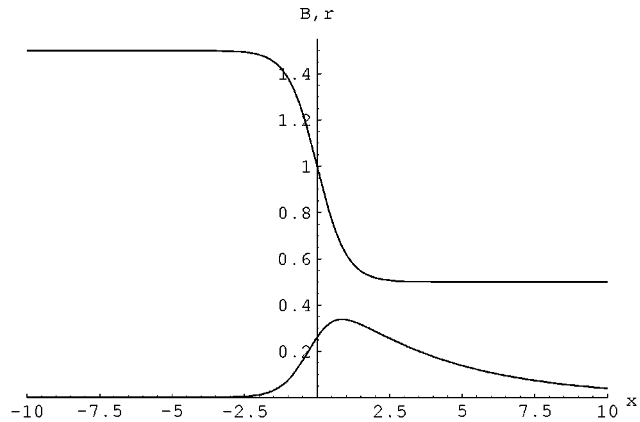


FIG. 1. Distribution of the $z$-component of the magnetic field and of the plasma density for $\alpha=0.25$, $\beta=0.5$, $B_1=1.5$, and $B_2=2.5$.

$$\beta=\frac{8\pi n_0 T}{B_0^2}=\frac{2c_s^2}{v_a^2\Gamma}. \tag{7}$$

Here $B_0$ is the reference magnitude of the magnetic field, $n_0=\rho_0/m_i$, $c_s$ is the sound speed, $\Gamma$ is the adiabatic index, $M=V/c_s$ is the Mach number, $v_a=B_0/(4\pi\rho_0)^{1/2}$ is the reference Alfvén velocity, Kn the Knudsen number defined as $\mathrm{Kn}=c_s/\nu^{(in)}L$ and $L$ is a scale length. The plasma velocity is equal to $V=\alpha c_s\mathrm{Kn}$.

We take the magnetic field to be of the form

$$B_z(X)=\frac{B_1+B_2}{2}+\frac{B_1-B_2}{2}\tanh\left(\frac{X}{L}\right), \tag{8}$$

where $B_1$ and $B_2$ are the magnetic field values ahead of and behind the plasma slab with $B_0=(B_1+B_2)/2$. We see that the magnetic field has a form of the shock wave that propagates with constant velocity $V$ along the $x$ axis.We consider boundary conditions such that the density $\rho$ vanishes at $X\to-\infty$.

In Fig. 1 the solution for $\alpha=0.25,\beta=0.5,B_1=1.5,B_2=2.5$ is shown. We see that the density distribution is characterized by two scale lengths, the shorter is equal to the magnetic field scale-length while the longer is about $1/\beta$ times larger.

## III. INTERCHANGELIKE INSTABILITY OF A THIN PLASMA SLAB

Now we show that the stationary solution obtained in the previous section is unstable. This instability is similar to the interchange instability of a fluid plasma supported against gravity by a magnetic field [6] and to the Rayleigh-Taylor instability. We shall analyze this instability in the long-wavelength approximation, when the perturbation wavelength is much larger than the slab width. In this approximation the plasma distribution can be assumed to have the form of a thin foil. We adopt the approach developed in Ref. [9] in the case of a plasma without a neutral component.

We assume that the foil (infinite in the $y$ and $z$ directions) is initially located in the $x=0$ plane. We take its thickness $l\to 0$ at constant surface mass density $\sigma=\rho l$. We consider a

2D case where all variables depend on two coordinates, $x$ and $y$, and on time $t$ and the foil moves in the $x$-$y$ plane.

We introduce the Lagrange variables, $x_0$, and $y_0$, related to the Euler coordinates by

$$x=x_0+\xi_x(x_0,y_0,t), \tag{9}$$

$$y=y_0+\xi_y(x_0,y_0,t). \tag{10}$$

Here $\xi_x(x_0,y_0,t)$ and $\xi_y(x_0,y_0,t)$ are the components of the foil displacement vector. We consider two points on the foil initially separated by the distance $ds_0=|d\mathbf{r}_0|$; at time $t$ they are separated by the distance $ds=|d\mathbf{r}|$, where

$$ds_0=[(dx_0)^2+(dy_0)^2]^{1/2} \tag{11}$$

and

$$ds=[(dx)^2+(dy)^2]^{1/2}, \tag{12}$$

respectively. In the Lagrange variables we have for the surface mass

$$\sigma_0(s_0)ds_0=\sigma(x_0,y_0,t)ds. \tag{13}$$

We introduce the surface mass Lagrange variable $m$, which is given by equation

$$m=\int^{s_0}\sigma_0(s_0)ds_0. \tag{14}$$

The magnetic field pressure acts on an element of the foil of length $ds=|d\mathbf{r}|$ with the force

$$d\mathbf{f}=\mathcal{P}(d\mathbf{r}\times\mathbf{e}_z), \tag{15}$$

where $d\mathbf{r}$ is a vector in the $x$-$y$ plane directed along the foil and

$$\mathcal{P}=\frac{B^2}{8\pi}. \tag{16}$$

The magnetic field $B$ is assumed to vanish at the front of the foil and to have a constant magnitude along the back of the foil. In the long-wavelength approximation, which we use in this paper, the jump of the magnetic field is constant since the electric current density, integrated across the foil cannot change its magnitude along the foil because of the condition div $\mathbf{J}=0$. The magnetic field pressure $\mathcal{P}=\mathcal{P}(t)$ is taken to be a given function of time.

Setting $d\mathbf{f}=dm\,\partial_t\mathbf{r}$, we obtain the equations of motion

$$\nu^{(in)}\sigma ds\dot{\mathbf{r}}=\mathcal{P}(d\mathbf{r}\times\mathbf{e}_z). \tag{17}$$

Writing $d\mathbf{r}=(d\mathbf{r}/ds_0)ds_0=(d\mathbf{r}/dm)dm$ we obtain the equations of motion of the foil in the surface-mass Lagrange variable

$$\partial_\tau x=\partial_m y, \tag{18}$$

$$\partial_\tau y=-\partial_m x, \tag{19}$$

where we have introduced the normalized time variable

$$\tau=\int^t\frac{\mathcal{P}(t)}{\nu^{(in)}}dt. \tag{20}$$

The stationary motion of the plasma slab with constant velocity along the $x$ axis corresponds to the solution of Eqs. (18) and (19):

$$x=\tau,\quad y=m. \tag{21}$$

This solution is unstable against perturbations of the form $(x-\tau,\ y-m)\propto\exp(iqm)$, where $q$ is the wave number of perturbations in Lagrangian variables. The growth rate of the instability is

$$\gamma=q. \tag{22}$$

The expression for the time dependence of the perturbations can be rewritten in dimensional units in the form $(x,\ y)\propto\exp[\int^t\gamma(t)dt]$ with $\gamma=gq/\nu^{(in)}$ where the effective gravity $g$ is equal to $g=v_a^2(t)/l$, $l$ is the foil thickness and $v_a=B/(4\pi\rho)^{1/2}$ is the Alfvén velocity. The growth rate of the Rayleigh-Taylor instability of a thin slab in the absence of friction is equal to $\gamma_{RT}=(gq)^{1/2}$ [9]. (This result applies also to a finite-width, internally homogeneous, slab with sharp boundaries [12]). We see that the ion-neutral collisions slow down the instability when the collision frequency is larger than $\gamma_{RT}$.

The solution of Eqs. (18), (19), with initial conditions $x(0)=\xi_{x0}\sin qm$ and $y(0)=m+\xi_{y0}\cos qm$, gives

$$x=\tau+[\xi_{x0}\cosh(q\tau)-\xi_{y0}\sinh(q\tau)]\sin qm, \tag{23}$$

$$y=m+[\xi_{y0}\cosh(q\tau)-\xi_{x0}\sinh(q\tau)]\cos qm, \tag{24}$$

with $\xi_{x0}$ and $\xi_{y0}$ the perturbation amplitudes. It describes the superposition of a uniform motion along the $x$ axis with constant velocity $V=1$ and of exponentially growing perturbations with wavelength $2\pi/q$. In the nonlinear stage of the instability, as in the case investigated in Ref. [9], the foil, initially located in the plane $x=0$, is folded and cusps and bubbles form with a periodic chain of maxima and minima along $y$. The density of the plasma in the foil is given by

$$\sigma=\frac{\sigma_0(m)}{[(\partial_m x)^2+(\partial_m y)^2]^{1/2}}. \tag{25}$$

At the top of the bubbles the density decreases exponentially in time: for $\tau\to\infty$ we obtain $\sigma\sim\sigma_0(q\xi_0)^{-1}\exp(-\gamma\tau)$. On the contrary, in the cusp region the density increases and a singularity appears in a finite time after which the solution cannot be continued. This singularity has been discussed in Refs. [9] and [10] and corresponds to the compression wave breaking.

## IV. USE OF THE CONFORMAL MAPPING TO DESCRIBE THE NONLINEAR STAGE OF THE RAYLEIGH-TAYLOR INSTABILITY

Equations (23) and (24) give only a particular solution of Eqs. (18) and (19). In order to find the generic solution of the Cauchy problem we observe that Eqs. (18) and (19) are simply the Cauchy-Riemann conditions for the real and imaginary parts of an analytical function $W(\zeta)$ of a complex variable

$$\zeta=m+i\tau. \tag{26}$$

The real part of $W(\zeta)$ is equal to the $x$ coordinate of the foil, while the $y$ coordinate is the imaginary part. Thus we write

$$x+iy=W(\zeta). \tag{27}$$

This expression is the conformal mapping from the complex plane $m+i\tau$ to the plane $x+iy$.

Following Ref. [13], where the nonlinear stage of the tearing mode of a thin current sheet was investigated, we notice that the analytical function $W(\zeta)$ is defined by its behavior on the real axis $t=0$, i.e., by the initial conditions $\xi_x(y_0,0)$ and $\xi_y(y_0,0)$. This gives the solution for the Cauchy problem for the elliptical system of Eqs. (18) and (19). According to Eq. (25) the surface mass of the foil is equal to

$$\sigma=\frac{\sigma_0(m)}{|W'(\zeta)|}, \tag{28}$$

where a prime denotes differentiation with respect to the complex variable $\zeta$.

The function $|W'(\zeta)|$ is the Jacobian of the transformation that gives the mapping of the curve $x+iy=W(m)$, the foil shape at $\tau=0$, to the curve $x+iy=W(m+i\tau)$, which describes the change of the foil shape. Here the time $\tau$ is the parameter of the mapping.

In order to show the typical behavior of the solution described by Eq. (27) we consider several different initial conditions.

(i) The initial conditions

$$x(m,0)=\kappa_R\frac{1}{1+m^2},\quad y(m,0)=-m+\kappa_I\frac{1}{1+m^2} \tag{29}$$

correspond to the analytical function

$$W(\zeta)=-i\zeta+\kappa\frac{1}{1+\zeta^2}, \tag{30}$$

where $\kappa=\kappa_R+i\kappa_I$ is a complex constant and its absolute value gives the amplitude of the perturbation. From Eq. (27) we find

$$x(m,\tau)=\tau+\kappa_R\frac{1-\tau^2+m^2}{(1-\tau^2+m^2)^2+4m^2\tau^2}+\kappa_I\frac{2m\tau}{(1-\tau^2+m^2)^2+4m^2\tau^2}, \tag{31}$$

$$y(m,\tau)=-m+\kappa_I\frac{1+\tau^2+m^2}{(1-\tau^2+m^2)^2+4m^2\tau^2}-\kappa_R\frac{2m\tau}{(1-\tau^2+m^2)^2+4m^2\tau^2}. \tag{32}$$

These expressions describe the growth of perturbations that are faster than exponential. It is easy to see that at the finite time $\tau=1$ the Jacobian $|W'|$ of the transformation becomes infinite at the point $m=0$. This corresponds to the rarefaction wave break that occurs with the formation of a hole in the plasma density distribution. In the case of a small amplitude perturbation. i.e., for a small absolute value of the parameter $\kappa$, we obtain from Eq. (30) in the neighborhood of $\zeta=i$,

$$W(\zeta)\approx-i\zeta+\kappa\frac{1}{2(\zeta-i)} \tag{33}$$

and

$$W'(\zeta)\approx-i-\kappa\frac{1}{2(\zeta-i)^2}. \tag{34}$$

As mentioned above, the Jacobian $|W'|$ tends to infinity for $\tau=1$ at point $m=0$ where the plasma density vanishes. For real positive (negative) $\kappa$ this singularity is accompanied by the compression wave breaks at $\tau=1\pm(\mp)|\kappa_R|^{1/2}/2$ at the points $m=\pm|\kappa_R|^{1/2}/2$, respectively, where the plasma density tends to infinity.

If $\kappa$ is a positive imaginary number, the singularity corresponds to the compression wave break at the point $m=0$ at time $t=1-(\kappa_I/2)^{1/2}$, while, if $\kappa$ is a negative imaginary number, the singularity corresponds to the compression wave break at the point $m=\pm(|\kappa_I|/2)^{1/2}$ at time $t=1$. The typical singularity corresponds to the case when both real and imaginary parts of $\kappa$ do not vanish. In this case the compression wave break occurs first, followed at $t=1$ by the rarefaction wave break at $m=0$. This case is illustrated in Fig. 2 for $\kappa=0.2(1+i)$.

(ii) A perturbation localized in a finite region can be described by the initial conditions

$$x(m,0)=\kappa_R\exp(-m^2),\quad y(m,0)=-m+\kappa_I\exp(-m^2). \tag{35}$$

The corresponding analytical function given by Eq. (27) is

$$W(\zeta)=-i\zeta+\kappa\exp(-\zeta^2), \tag{36}$$

where again $\kappa=\kappa_R+i\kappa_I$ is a complex constant. From Eq. (27) we find

$$x(m,\tau)=\tau+\exp(\tau^2-m^2)[\kappa_R\cos(2m\tau)-\kappa_I\sin(2m\tau)], \tag{37}$$

and

$$y(m,\tau)=m+\exp(\tau^2-m^2)[\kappa_I\cos(2m\tau)+\kappa_R\sin(2m\tau)]. \tag{38}$$

These expressions describe perturbations that grow faster than exponential. We see that in a finite time, $\tau\approx(\ln 1/2|\kappa|)^{1/2}$ for $|\kappa|\ll 1$, a compression wave break occurs with the formation of singular regions where the plasma density tends to infinity. In the regions in between, ''bubbles'' form where the plasma density decreases as $\exp(-\tau^2)$.

(iii) Now we consider the analytical function

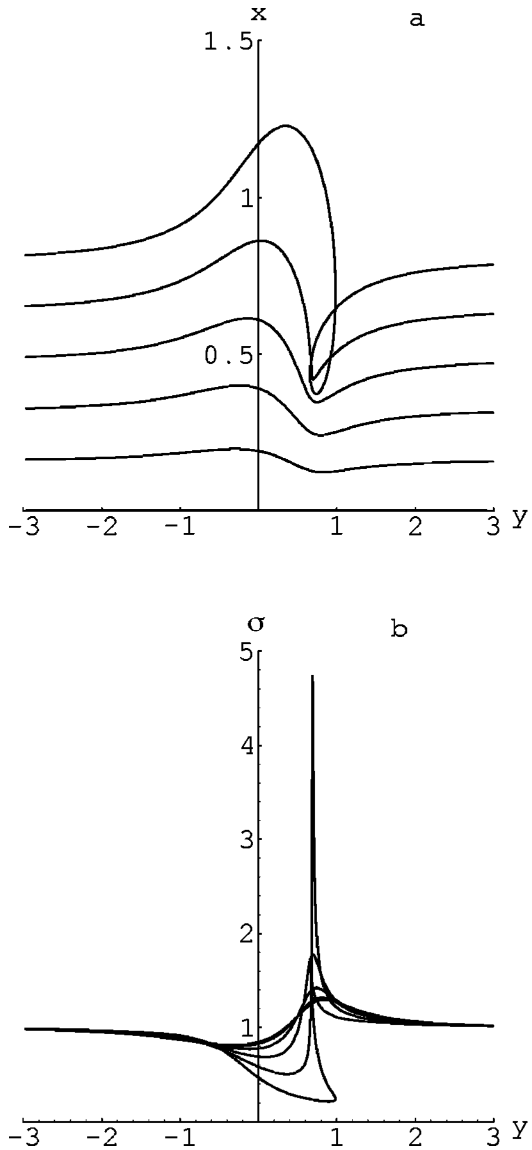


FIG. 2. Formation of the singularity on the foil in the case when neither the real nor the imaginary parts of $\kappa$ vanish: $\kappa=0.2(1+i)$. (a) Change of the foil shape and (b) density of the plasma versus the $y$ coordinate at $t=0$, 0,25, 0.5, 0.75, 0.9.

$$W(\zeta)=\exp(\pm i\zeta)+\kappa w(\zeta), \tag{39}$$

where $\kappa$ is supposed to be small and $-\pi\leqslant m\leqslant\pi$. This expression provides a solution of Eqs.(18) and (19) that describes the Rayleigh-Taylor instability of an initially nonplanar, thin shell. The shell has a circular form modulated in the azimuthal direction according to the function $\kappa w(\zeta)$. Depending on the sign in the exponent the shell either collapses or expands and its radius decreases or increases exponentially.

For initial conditions corresponding to an exponential modulation $w(m)=\exp(-iqm)$, with integer wave number $q>1$, we have

$$W(m+i\tau)=\exp(\mp\tau\pm im)+\kappa\exp(q\tau-iqm). \tag{40}$$

This expression describes a collapsing or expanding cylindrical shell, modulated with azimuthal number $q$ and initial modulation amplitude $|\kappa|$. We see that both converging and diverging shells are unstable against perturbations of the form given by Eq. (40). After a finite time interval, a compression wave breaking occurs with a periodic azimuthal structure. According to Eq. (40), in the Lagrange coordinates we have a linear superposition of the expressions that describe the cylindrical shell motion and the exponentially growing perturbations. In particular, there is no stabilization of the Rayleigh-Taylor instability due to stretching of the expanding shell. Similar patterns of the wave breaking in the converging and expanding shells are also seen in Figs. 3(a) and 3(b), which present the evolution of the instability in the framework of the Ott's model [9], which we discuss below.

According to heuristic arguments in Euler coordinates, stretching should saturate the perturbations when the shell radius $R$ increases. It is reasoned that in the Euler coordinates the wavenumber and the amplitude of the perturbations change according to

$$\dot{q}=-\frac{\dot{R}}{R}q, \quad \text{and} \quad \dot{\mathcal{W}}-q(\tau)\mathcal{W}=0. \tag{41}$$

For an exponentially growing radius of the shell, $R=R_0\exp(\tau)$, we find $q=q_0\exp(-\tau)$. In this case the heuristic reasoning predicts that the amplitude $\mathcal{W}(\tau)$ saturates. Perturbations of the converging shell instead grow faster than exponential due to the shell shrinking. Stabilization due to stretching has been discussed in Ref. [14], in the case of the vortex stability, and in Refs. [15–17] for the tearing mode of a current sheet.

Contrary to this prediction we see from Eq. (40) that, in the Lagrange coordinates, the perturbation wave number does not change. Indeed, in the Euler coordinates the typical scale length of perturbations decreases, which implies that the wave number grows. Since the wave number corresponds to the derivative with respect to the coordinates, using the relationship $d/ds=J^{-1}d/ds_0$, we can write the relation between the wave number in the Euler coordinates $q_E$ and that in the Lagrange coordinate $q_L$ as $q_E=J^{-1}q_L$. Here $J=|ds/ds_0|$ is the Jacobian of the transformation from the Euler to the Lagrange coordinates. If we linearize the expression for the Jacobian, assuming that the perturbations in Eq. (40) are relatively small, we obtain that the Jacobian increases exponentially for an expanding shell so that $q_E\propto\exp(-t)\rightarrow 0$. Nevertheless, nonlinear effects due to the perturbations make the Jacobian vanish when the compression wave break occurs. Thus the Rayleigh-Taylor instability is not stabilized either in the Lagrange or in the Euler coordinate plane.

However, in the case of an expanding shell, we must take into account the slowing down of the instability because the magnetic field inside the shell, and thus the magnetic pressure acting on the shell, decrease. Assuming that the amplitude of the shell modulation is smaller that the shell radius, we estimate the magnetic flux inside the cylindrical shell as

$$BR^2=\Phi=\text{const}. \tag{42}$$

Then, for $R(\tau)=R_0\exp(\tau)$, we find from Eq. (20)

$$\tau=\tfrac{1}{4}\ln\left(\frac{\Phi^2}{2\pi R_0^4\nu^{(in)}}t\right). \tag{43}$$

Inserting this expression for $\tau$ into Eq. (40) we obtain that the shell radius increases proportionally to $t^{1/4}$ while the perturbations grow as $t^{q/4}$.

We shall further discuss the effects of the shell expansion, or compression, later in the context of Ott's problem of the Rayleigh-Taylor instability of a thin shell.

## V. LOCAL STRUCTURE OF THE WAVE BREAK IN STABLE AND UNSTABLE MEDIA

We have described several scenarios of wave breaking that correspond to the so-called ''gradient catastrophe'' when the gradients in the perturbations tend to infinity after a finite time interval. All these singularities can be subdivided into two classes according to the evolution of the Jacobian of the transformation from the Lagrange to the Euler coordinates. In our case the Jacobian is equal to $|W'(\zeta)|$. The the first kind of wave break corresponds to the compression case where the Jacobian vanishes at the point $\zeta_0$. This singularity is typical for both stable and unstable media. The discussion of this wave break is presented in Refs. [18] and [19]. In collisionless media the compression wave break results in the self-intersection of the particle trajectories and in the formation of regions with the multistream motion. In dissipative media, in the 1D case, compression wave break results in the formation [18] of a shock wave. In the second kind of singularities, which do not exist in stable media, the Jacobian becomes infinite at the critical point of the transformation $\zeta_0$. This singularity corresponds to the rarefaction wave break.

In the case of the break of the first type, with $|W'(\zeta_0)| = 0$, we can expand the function $W'(\zeta)$ in the vicinity of the critical point $\zeta_0$ which, with a coordinate change, can be shifted to the origin. Thus we write

$$W'(\zeta) = i\alpha\zeta + i\beta\zeta^2 + \cdots \qquad (44)$$

and

$$W(\zeta) = i\alpha\frac{\zeta^2}{2} + i\beta\frac{\zeta^3}{3} + \cdots, \qquad (45)$$

where $\alpha$ and $\beta$ are complex constants (by a shift in the coordinates this dependence can be transformed into the standard form of a cubic curve used in the theory of singularities [20]: $i\tilde{\alpha}\tilde{\zeta} + i\tilde{\beta}\tilde{\zeta}^3 + \cdots$).

The essential property of the local representation of the mapping from the Lagrange variables to the Euler variables given by Eq. (45) is the cubic dependence on the $m$ coordinate. Modulo a rotation in the $x$-$y$ plane, $\beta$ can be taken real and positive. Then we have

$$x = \beta\frac{\tau^3}{3} + \alpha_I\frac{\tau^2}{2} - \alpha_R\tau m - \left(\beta\tau + \frac{\alpha_I}{2}\right)m^2 \cdots, \qquad (46)$$

$$y = -\alpha_R\frac{\tau^2}{2} - (\beta\tau^2 + \alpha_I\tau)m + \frac{\alpha_R}{2}m^2 + \frac{\beta}{3}m^3 \ldots, \qquad (47)$$

where only $y$ has a cubic dependence on $m$. At the critical time $\tau = 0$ the foil has a cusplike shape with sides given by

$$y = -\left(\frac{\alpha_R}{\alpha_I}\right)x \pm \frac{\beta}{3}\left(\frac{2}{\alpha_I}\right)^{3/2} x^{3/2} \quad \text{and} \quad \frac{x}{\alpha_I} > 0. \qquad (48)$$

If $\alpha_R = 0$ the cusp axis is aligned along the $x$ axis. The case $\alpha_I = 0$ is exceptional and leads to a different cusp form. The singularity described by Eq. (45) has a one-dimensional structure and the other dimensions ''dress'' it according to Ref. [20]. Considering for the sake of simplicity the case $\alpha_R = 0$, we see that the singularity appears along the $y$ coordinate which, if $\alpha_I > 0$, is a single valued function of $m$ for $\tau < 0$ and becomes multivalued for $\tau > 0$.

Now we discuss the break of the second kind, when $|W'(\zeta_0)| = \infty$. Again, the critical point $\zeta_0$ can be shifted to the origin. In order to analyze the local structure of this mapping, we consider the inverse mapping $U(z) = W^{-1}(z)$ from the plane $x + iy$ to the plane $m + i\tau$ :

$$m + i\tau = U(z), \qquad (49)$$

with $z = x + iy$. According to the rule of differentiation of the inverse functions we find

$$\frac{dU}{dz} = \frac{1}{\dfrac{dW}{d\zeta}}, \qquad (50)$$

which proves the known property that the Jacobian of the inverse mapping vanishes, $U'(z_0) = 0$, at the critical point of the direct transformation where the Jacobian $W'(\zeta_0) = \infty$. The coordinates of the critical point in the $x + iy$ plane are given by $z_0 = W(\zeta_0)$.

The behavior of the system near the singularity is different depending on whether the value of $z_0 = W(\zeta_0)$ is finite or infinite.

If the position of the singularity is at a finite distance in the $x$-$y$ plane from the initial foil position, we can repeat the argument given above Eq. (44) and obtain the local expansion

$$U(z) = i\frac{\alpha'}{2}z^2 + i\frac{\beta'}{3}z^3 + \cdots, \qquad (51)$$

which is the counterpart of Eq. (45). We note that Eq. (51) corresponds to a solution of Eqs. (18) and (19), where we have performed the hodograph transformation that interchanges dependent and independent variables. This transformation yields

$$\partial_x\tau = \partial_y m, \qquad (52)$$

$$\partial_y\tau = -\partial_x m. \qquad (53)$$

In the $x$-$y$ plane relationship (51) corresponds to the expansion

$$W(\zeta) = \left(\frac{2i}{\alpha'}\right)^{1/2}\zeta^{1/2} - \left(\frac{-2i\beta'^2}{9\alpha'^3}\right)^{1/2}\zeta^{3/2} + \cdots. \qquad (54)$$

Choosing now, for the sake of convenience, both $\alpha'$ and $\beta'$ real we obtain instead of Eqs. (46) and (47)

$$m = -\alpha' xy - \beta' x^2 y + \frac{\beta'}{3} y^3 \cdots, \tag{55}$$

$$\tau = \frac{\alpha'}{2}(x^2 - y^2) + \frac{\beta'}{3} x^3 - \frac{\beta'}{3} xy^2 \cdots. \tag{56}$$

Equation (56) shows that at $\tau \approx 0$ this break occurs along hyperbolic arc segments. One branch of hyperbola, describing the local shape of the foil before the break, splits at $x = 0, y = 0$ into two hyperbolic segments moving in the two neighboring quadrants.

When the singularity in the $x$-$y$ plane is at infinity, we use the local expansion

$$W(\zeta) = -i\zeta - i\frac{\beta}{\zeta} + \cdots, \tag{57}$$

which corresponds to Eq. (34). For $U(z)$ in the limit $z \to \infty$ we have $U(z) = \zeta = -i\beta/2z$. Separating real and imaginary parts, we find that asymptotically the form of the shell near the singularity is described by a circle,

$$\left(x + \frac{\beta}{2\tau}\right)^2 + y^2 = \frac{\beta^2}{4\tau^2}. \tag{58}$$

Both the radius and the center of the circle tend to infinity for $\tau \to 0$. A similar evolution of the perturbations is seen in Fig. 2.

## VI. THE ILL POSEDNESS OF THE PROBLEM OF A THIN SHELL INSTABILITY

The expressions obtained above describe the evolution of initially infinitesimal perturbations of foils with a smooth shape for $\tau < \tau_0$, where $\tau_0$ is the time when the singularity appears. These solutions cannot be continued after the singularity. The singularities are formed when the Jacobian of the transformation from the Euler to the Lagrange coordinates $|W'(\zeta)|$ either vanishes or becomes infinite. The latter case corresponds to a pole or to a cut of the analytical function $W(\zeta)$. An arbitrary small change of the form of the initial perturbation can lead to the appearance of a new pole in the complex plane $m + i\tau$ and can change the time and the location of the singularity. This is a well known property of the ill-posed problems for partial differential equations [19]. In connection with the discussion of the final stage of various instabilities, the ill posedness has been discussed in Refs. [21,22,14], in the case of the Kelvin-Helmholtz instability of vortex sheets, and in Ref. [13] in the case of the tearing instability of the current sheet.

We notice that the nonlinear interactions do not regularize the solutions, contrary to the hypothesis made in Ref. [14], as nonlinear interactions themselves lead to the appearance of singularities.

There are two reasons for the formation of a singularity after a finite time. The first mechanism is the compression wave break, which is inherent to nonlinear systems and originates from the $(\mathbf{v}\cdot\nabla)\mathbf{v}$ term in the hydrodynamic equations. The second reason is due to the fact that the problems under consideration are ill posed and it originates from the use of the long-wavelength approximation. The expressions that are derived in this approximation lead to instability growth rates that increase with the wave number of the perturbations. If the Fourier transform of the analytical function $W(\zeta)$ of the complex variable $\zeta = m + i\tau$, taken at $\tau = 0$, gives a function $\tilde{W}(q)$ that extends to infinity, it leads unavoidably to a singular behavior of the function in the complex plane $m + i\tau$ at a finite distance from the origin. In other words, there are disturbances which grow arbitrarily fast when $q \to \infty$. Instead, if the Fourier transform $\tilde{W}(q)$ is nonzero only inside a finite size region $|q| < q_m$, and vanishes identically beyond this region, the analytical function $W(\zeta)$, obtained by inverse Fourier transform, has a singular point only at infinity [23].

Outside the domain where the long-wavelength approximation holds, we expect that the growth rate saturates as a function of the wave number $q$ and vanishes either at infinity or at a finite value $q = q_m$. This provides a regularization of the solution. To show this regularization we consider the one pole approximation of the singularity given by Eq. (33). The Fourier transform of the function $W(\zeta) = i\zeta + a/\zeta$ is the sum of the derivative of a $\delta$ function and of a Heaviside step function: $\tilde{W}(q) = -\delta'(q) - ia\theta(q)$, which extends to infinity in the $q$ coordinate. In order to regularize this solution we consider a heuristic argument and suppose that the short-wavelength behavior of the instability results in the decay of the perturbations with $q > q_m$. This decay can be taken into account as a cutoff of the function $\tilde{W}(q)$ at $q_m$ by writing $\tilde{W}(q) = -\delta'(q) - ia\theta(q)\theta(q_m - q)$. The inverse Fourier transform of this function gives $W(\zeta) = i\zeta + a(1 - \exp(iq_m\zeta))/\zeta$. For $|\zeta| \gg 1/q_m$ the behavior of this function is similar to that in the one pole approximation. Instead, when $|\zeta| < 1/q_m$ the growth of perturbations slows down and becomes exponential. In addition, a modulation of the solution appears, with wavelength $2\pi/q_m$.

Before concluding this section it is important to observe that these limitations, which are intrinsic to the long-wavelength approximation, have different consequences in the case of the rarefaction and of the compression wave break. This difference arises from the opposite behavior of the characteristic perturbation wave number $k(t)$ when the singularity is approached in the two cases. The long-wavelength limit corresponds to $kL \ll 1$, where $L$ the thickness of the shell. According to the analysis of Sec. IV, the absolute value at time $t$ of the wave number $k(t)$ is inversely proportional to the Jacobian $|W'(\zeta)|$ in Eq. (28).

In the case of the rarefaction wave break the Jacobian $|W'|(t \to t_0) \to \infty$ so that the long-wavelength approximation works better and better close to the singularity since $k$ tends to zero as the singularity is approached. Futhermore, if we assume that the shell keeps a constant volume density during its evolution, Eq. (28) shows that close to the rarefaction singularity the stretching of the shell results in a corresponding reduction of its width, which also contributes to the validity of the long-wavelength approximation.

Different is the case of the compression break since $|W'|(t \to t_0) \to 0$. In this case, after a relatively short time, the solution enters the short-wavelength regime and its further evolution depends on the specific details of the internal structure of the plasma slab, which can no longer be considered as a thin shell. When the wavelength of the nonlinear perturbation becomes much shorter than the slab width, and the slab has sharp boundaries, we may adopt the semi-infinite domain model which is well developed in the case of the Rayleigh-Taylor instability starting from the work by Fermi and von Neumann [25]

## VII. OTT'S PROBLEM FOR AN INITIALLY NONPLANAR SHELL

In this section we assume that the density of the neutral component vanishes so that the equation for the foil motion is $\sigma ds \ddot{\mathbf{r}} = d\mathbf{f}$ . The evolution of the Rayleigh-Taylor instability of an initially nonplanar thin shell was investigated numerically in Refs. [10] and [11], while an azimuthally symmetric configuration where the variables depend on the $r$ and $z$ coordinates was discussed in Ref. [24]. Here we consider the case of an azimuthally asymmetric foil which is taken to be uniform along the $z$-axis. In the mass Lagrange coordinates the 2D equations of motion of the foil, Eqs.(2) and (3) of Ref. [9], take the form

$$\partial_{\tau\tau} x = \partial_m y, \tag{59}$$

$$\partial_{\tau\tau} y = -\partial_m x. \tag{60}$$

From these equations it follows that the complex function $w(m,\tau) = x + iy$ obeys equation

$$\partial_{\tau\tau} w = -i\partial_m w, \tag{61}$$

while for the complex conjugate function $w^*(m,\tau) = x - iy$ we have

$$\partial_{\tau\tau} w^* = i\partial_m w^*, \tag{62}$$

where $w$ and $w^*$ are considered as independent functions with $x = (w + w^*)/2$, and $y = -i(w - w^*)/2$.

The particular solution $w = im + \tau^2/2$ corresponds to a uniformly accelerated plane foil while the solution

$$w(m,\tau) = x + iy = im^3 - i\tfrac{1}{4} m\tau^4 - \tfrac{1}{120}\tau^6 + \tfrac{3}{2} m^2\tau^2 \tag{63}$$

describes the local structure of the wave breaking and has the characteristic cubic dependence of the coordinate $y$ on $m$.

For $w(m,\tau) \propto \exp(imq)$ Eq. (61) describes exponentially growing and decaying modes for $q > 0$ and oscillatory modes with real frequency for $q < 0$. The intervals in $q$ are interchanged in Eq. (62) for $w^*(m,\tau) \propto \exp(imq)$.

For initial conditions $w_0(m) = w_1 \exp(im) + w_3 \exp(iqm)$ and $\partial_\tau w_0(m) = w_2 \exp(im) + w_4 \exp(iqm)$ and $w_0^*(m) = w_5 \exp(-im) + w_6 \exp(-iqm)$ and $\partial_\tau w_0^*(m) = w_7 \exp(-im) + w_8 \exp(-iqm)$ the solutions of Eqs. (61) and (62) are of the form

$$w(m,\tau) = (w_1 \cosh\tau + w_2 \sinh\tau)\exp(im) + (w_3 \cosh\sqrt{q}\tau + w_4 \sinh\sqrt{q}\tau)\exp(iqm) \tag{64}$$

and

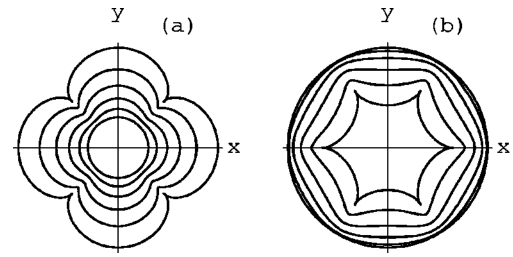


FIG. 3. Development of the Rayleigh-Taylor instability in the case of Ott's problem (a) for a radially expanding and (b) for a converging cylindrical shell.

$$w^*(m,\tau) = (w_5 \cos\tau + w_6 \sin\tau)\exp(-im) + (w_7 \cos\sqrt{q}\tau + w_8 \sin\sqrt{q}\tau)\exp(-iqm) \tag{65}$$

with constant $w_\alpha$.

We consider an initial configuration with $q$ an integer greater than one, $w_1 = w_2 = w_3 = 0$ and $w_4 = \kappa \ll 1$, and $w_5 = 1, w_6 = w_7 = w_8 = 0$. In this case Eqs. (64) and (65) describe a collapsing cylindrical shell, modulated with azimuthal number $q$ and initial amplitude $\kappa$. As we can see in Fig. 3(a) the radially expanding shell is unstable against perturbations of the form given by Eq. (64). After a finite time interval, compression wave breaking occurs with periodic azimuthal structure.

A converging shell is also unstable, as we see in Fig. 3(b). The development of modes of this type was observed experimentally in Ref. [26], examining the Rayleigh-Taylor instability of a cylindrical slab. According to Eqs. (64) and (65), in the Lagrange coordinates we have a linear superposition of terms that describe the motion of the cylindrical shell and exponentially growing perturbations. As in the case studied in Sec. IV, there is no stabilization of the Rayleigh-Taylor instability due to stretching of the expanding shell. If we take into account that the pressure inside the shell decreases, we find again that the exponential dependence of the perturbations and of the expansion of the shell on the variable $\tau$ corresponds to slower, algebraic dependences on time.

Now we consider the generic solution of Eq. (61). This equation admits seven symmetry transformations represented by the operators:

$$X_\infty = w_1(m,\tau)\partial_w, \tag{66}$$

where $w_1(m,\tau)$ is a solution of Eq. (61),

$$X_1 = i\partial_m, \tag{67}$$

$$X_2 = \partial_\tau, \tag{68}$$

$$X_3 = 2m\partial_m + \tau\partial_\tau, \tag{69}$$

$$X_4 = 2im\partial_\tau - \tau w\partial_w, \tag{70}$$

$$X_5 = i\tau m\partial_\tau - im^2\partial_m - \tfrac{1}{4}(\tau^2 + 2im)w\partial_w, \tag{71}$$

$$X_6 = w\partial_w. \tag{72}$$

A discussion of the theory of the Lie group analysis of differential equations is presented in Refs. [27,28]. The operator $X_\infty$ stems from the fact that Eq. (61) is linear with respect to

$w(m,\tau)$ so that, to any solution $w(m,\tau)$, one can add any other solution $w_1(m,\tau)$. The operators $X_1$ to $X_4$ and $X_6$ correspond to the existence of stationary, of uniform solutions and to the invariance with respect to stretching of the variables, respectively. The operator $X_5$ represents the transformation

$$\overline{m}=\frac{m}{1-am}, \quad \overline{\tau}=\frac{\tau}{1-am},$$
$$\overline{w}=(1-am)^{1/2}\exp\left(\frac{ia\tau^2}{4(1-am)}\right)w. \tag{73}$$

Under this transformation a solution $f(m,\tau)$ takes the form

$$w(m,\tau)=\frac{1}{(1-am)^{1/2}}\exp\left(-\frac{ia\tau^2}{4(1-am)}\right)\times f\left(\frac{m}{1-am},\frac{\tau}{1-am}\right). \tag{74}$$

If we choose $f=i/(4\pi)^{1/2}$ and $a=-1/h$ in Eq. (74), and superpose $w=im+\tau^2/2$, we obtain a solution of the form

$$w(m,\tau)=im+\frac{\tau^2}{2}+\frac{w_0}{[4\pi(m+h)]^{1/2}}\exp\left(i\frac{\tau^2}{4(m+h)}\right), \tag{75}$$

where $w_0=i(h)^{1/2}$ is the initial perturbation amplitude and $h$ is a complex parameter. The shell is initially a planar foil with perturbations localized in a region with size of order $|h|$. If $h$ is imaginary and positive, $h=i|h|$, this solution describes perturbations that grow faster than exponential: $\propto\exp(\tau^2/4|h|)$. The typical singularity corresponds to the nonlinear superposition of compression and rarefaction waves. After a finite time the compression wave breaks, while it takes an infinite time for the rarefaction wave break to occur.

The superposition of solutions

$$w(m,\tau)=(w_1\cosh\tau+w_2\sinh\tau)\exp(im)+\frac{w_0}{[4\pi(m+h)]^{1/2}}\exp\left(i\frac{\tau^2}{4(m+h)}\right) \tag{76}$$

describes the growth of nonlinear perturbations on a cylindrical shell. We see that both converging and diverging shells are unstable.

## VIII. CONCLUSIONS AND DISCUSSION

We have analyzed the nonlinear magnetohydrodynamics of a weakly ionized plasma slab. We have demonstrated that the structure of the magnetic field in a stationary plasma slab, moving under the magnetic field pressure, is similar to that of a shock wave. In this case the magnetic field pressure is balanced by the friction due to the interaction with the neutrals. This configuration is unstable against a Rayleigh-Taylor-like instability. In the long-wavelength limit the plasma slab can be approximated as a thin shell. The nonlinear Cauchy problem for the thin shell has analytical solutions, which we have expressed in terms of analytical functions of a complex variable. We have shown that after a finite time a singularity forms. These singularities are of two types. (In the experiments on the nonlinear evolution of the Rayleigh-Taylor instability of thin shell these singularities are seen as bubbles and spikes [29]) The first type corresponds to the folding of the shell leading to the compression wave breaking. Mathematically, it arises from the nonlinear relationship between the Lagrange and the Euler coordinates. The second type corresponds to the tearing of the shell leading to the formation of a hole in the density. In this case the instability growth is faster than exponential. Ott's problem of the Rayleigh-Taylor instability of a thin nonplanar shell also has solutions with nonexponential growth of the perturbations. In addition, in the case of a nonplanar shell, stretching does not stabilize the perturbations. In the long-wavelength approximation, the tearing of the shell after a finite time and in general the faster than exponential growth are the result of the ill posedness of the Cauchy problem. Since this ill posedness is regularized outside the framework of the long-wavelength model, the regimes with nonexponential growth found in this paper are meant to describe the intermediate asymptotic behavior of the perturbations. However, in the case of the rarefaction wave break, the long-wavelength approximation holds up to the formation of the singularity since the perturbation wave number tends to zero as the singularity is approached.

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